Lecture 19: Discuss practice problems for Exam 2

#1: we first recall: §4.7 Lecture 12

Algorithm: find solns to Cauchy-Euler eqn

$$ax^2y'' + bxy' + cy = 0 \text{ for } x>0 \quad (*)$$

Step 1: Solve the characteristic eqn

$$ar^2 + (b-a)r + c = 0 \quad (1)$$

Step 2: $\Delta = (b-a)^2 - 4ac$

(I) If (1) has two distinct real roots r_1, r_2, ($\Delta > 0$)

then we have two linearly independent solns:

$$y_1 = x^{r_1}, \quad y_2 = x^{r_2}$$

(II) If (1) has only one repeated root r_0, ($\Delta = 0$)

then we have two linearly independent solns:

$$y_1 = x^{r_0}, \quad y_2 = x^{r_0}\ln x$$

(III) If (1) has complex roots $\alpha \pm \beta i$, ($\Delta < 0$)

$\beta \neq 0$, then we have two linearly independent solns:

$$y_1 = x^{\alpha}\cos(\beta \ln x), \quad y_2 = x^{\alpha}\sin(\beta \ln x)$$

In any of the cases (I), (II), (III), the general soln to (*) is

$$y = C_1 y_1 + C_2 y_2, \quad C_1, C_2 \in \mathbb{R}$$

#1: Solve the I.V.P

$$\begin{cases} x^2 y'' - 2xy' + 2y = 0 \\ y(1) = 2, \quad y'(1) = 3 \end{cases}$$

$$\Rightarrow \begin{cases} a = 1 \\ b = -2 \\ c = 2 \end{cases}$$

A: Step 1:

solve the associated characteristic eqn:

$$ar^2 + (b-a)r + c = 0$$

That is,

$$r^2 - 3r + 2 = 0$$

$$\Rightarrow \quad (r-1)(r-2) = 0$$

Two distinct roots: $r_1 = 1$, $r_2 = 2$.

Step 2:

Hence we get two linearly independent solns:

$$y_1 = x^{r_1} = x, \quad y_2 = x^{r_2} = x^2$$

The general solution:

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 x + c_2 x^2$$

$$y(1) = 2 \Rightarrow 2 = c_1 + c_2 \quad ①$$

$$y'(x) = c_1 + 2c_2 x$$

$$\underbrace{y'(1) = 3}_{\substack{x=1 \\ y'=3}} \Rightarrow 3 = c_1 + 2c_2 \quad ②$$

By ①, ② $\Rightarrow \begin{cases} c_1 = 1 \\ c_2 = 1 \end{cases}$

Hence the soln to the I.V.P is

$$y = x + x^2$$

#2: First recall §4.4, Lecture 9

Undetermined Coefficients

Thm: Consider the D.E

$$ay'' + by' + cy = C_0 x^m e^{rx} \qquad (4)$$

where $m \geq 0$ is an integer, r is a real number $C_0 \in \mathbb{R}$

(If $r=0$, then RHS of $(4) = C_0 x^m$)

Its associated characteristic/auxiliary eqn is

$$a\lambda^2 + b\lambda + c = 0 \qquad (5)$$

(I) If r is not a root of (5), then use the test function

$$y_p = (A_m x^m + A_{m-1} x^{m-1} + \cdots + A_1 x + A_0) e^{rx}$$

Here $A_m, A_{m-1}, \ldots, A_1, A_0$ are to be determined

(II). If r is a simple ($\Leftrightarrow$ not repeated) root of (5), then use the test function

$$y_p = x(A_m x^m + A_{m-1} x^{m-1} + \cdots + A_1 x + A_0) e^{rx}$$

(III) If r is a repeated root of (5), then use the test function

$$y_p = x^2 (A_m x^m + A_{m-1} x^{m-1} + \cdots + A_1 x + A_0) e^{rx}$$

Thm: Consider the D.E

$$ay'' + by' + cy = \begin{cases} C_0 x^m e^{\alpha x} \cos(\beta x) \\ \text{or} \\ C_0 x^m e^{\alpha x} \sin(\beta x) \end{cases}$$

Here $\begin{cases} a, b, c \in \mathbb{R} \\ m \geq 0 \text{ integer} \\ \alpha, \beta \in \mathbb{R} \end{cases}$ (Here $\beta \neq 0$)

Its char. eqn is

$$a\lambda^2 + b\lambda + c = 0 \qquad (6)$$

(I) If $\alpha \pm i\beta$ is not a root of (6),

then try test function

$$y = (A_m x^m + A_{m-1} x^{m-1} + \cdots + A_1 x + A_0) e^{\alpha x} \cos(\beta x)$$
$$+ (B_m x^m + B_{m-1} x^{m-1} + \cdots + B_1 x + B_0) e^{\alpha x} \sin(\beta x)$$

(II) If $\alpha \pm i\beta$ is a root of (6), then

try the test function

$$y = x(A_m x^m + A_{m-1} x^{m-1} + \cdots + A_1 x + A_0) e^{\alpha x} \cos(\beta x)$$
$$+ x(B_m x^m + B_{m-1} x^{m-1} + \cdots + B_1 x + B_0) e^{\alpha x} \sin(\beta x)$$

#2. Use the method of undetermined coefficients to solve the D.E. Find the general soln.

$$y'' - 2y' - 3y = \underbrace{2e^{3t}}_{C_0 t^m e^{rt}} \Rightarrow \begin{array}{l} C_0 = 2 \\ m = 0 \\ r = 3 \end{array}$$

A:

Step 1: Solve the associated homogeneous eqn:

$$y'' - 2y' - 3y = 0$$

Char. eqn: $\lambda^2 - 2\lambda - 3 = 0$

$$(\lambda - 3)(\lambda + 1) = 0$$

$$\Rightarrow \lambda_1 = 3, \ \lambda_2 = -1. \Rightarrow r = 3 \text{ is a simple root}$$

$$\Rightarrow y_1 = e^{3t}, \quad y_2 = e^{-t}$$

Step 2: Determine the format of the test function

Note $r=3$ is a simple root of the char. eqn

$$y_p = t \cdot Ae^{3t} = Ate^{3t}$$

Step 3: plug y_p into the eqn and find the value of A.

plug $y_p = Ate^{3t}$ in the D.E "$y''-2y'-3y=2e^{3t}$".

Compute: $y_p' = Ae^{3t} + 3Ate^{3t}$

$$y_p'' = 3Ae^{3t} + 3Ae^{3t} + 9Ate^{3t}$$
$$= 6Ae^{3t} + 9Ate^{3t}$$

$\Rightarrow$

$$LHS = y_p'' - 2y_p' - 3y_p$$

$$= (6Ae^{3t} + \cancel{9Ate^{3t}}) - 2(Ae^{3t} + \cancel{3Ate^{3t}})$$
$$\cancel{-3Ate^{3t}}$$
$$= 4Ae^{3t}$$

$RHS = 2e^{3t}$

$\Rightarrow \quad 4A = 2 \quad \Rightarrow A = \frac{1}{2}$

Hence $\quad y_p = \frac{1}{2} t e^{3t}$

Step 4. write down the general soln:

$$y = y_p + c_1 y_1 + c_2 y_2$$

$$\Rightarrow \quad y = \frac{1}{2} t e^{3t} + c_1 e^{3t} + c_2 e^{-t}$$

#3. First recall §4.6, Lecture 11.

Algorithm of variation of parameters method:

Find a particular soln to

$$a(x)\, y'' + b(x)\, y' + c(x)\, y = f(x) \quad (5)$$

Step 1: Find two linearly independent solns

$y_1(x)$, $y_2(x)$ to the homogeneous eqn

$$a(x)\, y'' + b(x)\, y' + c(x)\, y = 0$$

$a(x)$: the coefficient of y''

Step 2: Compute

$$V_1(x) = \int \frac{-f y_2}{a(x)\, W(y_1, y_2)}\, dx$$

$$V_2(x) = \int \frac{f y_1}{a(x)\, W(y_1, y_2)}\, dx$$

Here $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$ → Wronskian of y_1, y_2

Step3: $y_p = v_1 y_1 + v_2 y_2$

#3. Use variation of parameters to solve:

$$y'' + 4y = \sec(2t)$$

Find the general solns.

A:

Step 1: Solve the assoc. homogeneous eqn:

$$y'' + 4y = 0$$

char. eqn: $\lambda^2 + 4 = 0$

$\Rightarrow \lambda = \pm 2i = 0 \pm 2i$

$\alpha \pm \beta i$

$\alpha = 0, \ \beta = 2$

Hence $y_1 = \cos 2t \qquad y_2 = \sin 2t$

Seep2. Compute v_1, v_2

Note $\begin{cases} y_1 = \cos 2t \\ y_2 = \sin 2t \\ f = \sec 2t = \frac{1}{\cos 2t} \\ a(t) = 1 \end{cases}$

$$\Rightarrow \quad W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

$$= \cos 2t\,(2\cos 2t) - \sin 2t(-2\sin 2t)$$

$$= 2(\cos^2 2t + \sin^2 2t)$$

$$= 2$$

$$\Rightarrow \quad v_1 = \int \frac{-f y_2}{a(t)\, W(y_1, y_2)}\, dt$$

$$= \int \frac{-\frac{1}{\cos 2t} \sin 2t}{2}\, dt$$

$$= -\frac{1}{2} \int \frac{\sin 2t}{\cos 2t}\, dt$$

Use u-sub. $u=\cos 2t \Rightarrow du=-2\sin 2t\,dt$

$\Rightarrow \sin 2t\,dt = \frac{du}{-2}$

$$= \frac{1}{4}\int \frac{1}{u}\,du$$

$$= \frac{1}{4}\ln|u| + C$$

$$= \frac{1}{4}\ln|\cos 2t| + C$$

Choose $C=0 \Rightarrow v_1 = \frac{1}{4}\ln|\cos 2t|$.

$$v_2 = \int \frac{f\,y_1}{a(t)\,W(y_1,y_2)}\,dt$$

$$= \int \frac{\frac{1}{\cos 2t}\cos 2t}{1\cdot 2}\,dt$$

$$= \frac{1}{2}\int dt = \frac{t}{2} + C$$

Choose $C=0 \Rightarrow v_2 = \frac{t}{2}$

Step3: $y_p = v_1 y_1 + v_2 y_2$

$$= \frac{1}{4}\ln|\cos(2t)|\cdot\cos(2t) + \frac{t}{2}\sin 2t$$

Step 4: Write the general soln:

$$y = y_p + c_1 y_1 + c_2 y_2$$

$$= \frac{1}{4} \ln|\cos(2t)| \cdot \cos(2t) + \frac{t}{2} \sin 2t + c_1 \cos 2t + c_2 \sin 2t$$

★ • General solns to homogeneous eqn $ay'' + by' + cy = 0$ is
$$y = c_1 y_1 + c_2 y_2$$

• General solns to nonhomogeneous eqn $ay'' + by' + cy = f(x)$, $f(x) \neq 0$
is $y = y_p + c_1 y_1 + c_2 y_2$

---

4. Determine $\mathcal{L}^{-1}\left\{\dfrac{3s^2+5s+3}{s^4+s^3}\right\}$

A: Step 1. Find the partial fractional decomposition of $\dfrac{3s^2+5s+3}{s^4+s^3}$.

Note the denominator:

$$s^4 + s^3 = s^3(s+1)$$

$\Rightarrow$

$$\frac{3s^2+5s+3}{s^4+s^3} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+1}$$

We need to find the values of A, B, C, D

Multiple the above by $s^4+s^3 = s^3(s+1)$

$\Rightarrow$

$$3s^2+5s+3 = As^2(s+1) + Bs(s+1) + C(s+1) + Ds^3$$

↑ Lecture 15

plug in the roots of "$s^3(s+1)$"

① $s=0 \Rightarrow$

$3 = LHS = RHS = C \Rightarrow C = 3$

② $s=-1 \Rightarrow$

$1 = LHS = RHS = -D \Rightarrow D = -1$

plug some other "simple" values of s:

↓ $|s|$ cannot be large

③ $S=1 \Rightarrow$

$$11 = LHS = RHS = 2A + 2B + 2C + D$$

$$\Rightarrow 2A + 2B = 11 - 2C - D$$
$$= 6$$

$$\Rightarrow A + B = 3$$

③ $S=-2 \Rightarrow$

$$5 = LHS = RHS = -4A + 2B - C - 8D$$

$$\Rightarrow -4A + 2B = -5 - C - 8D$$
$$= 0$$

$$\Rightarrow -2A + B = 0$$

$$\Rightarrow \begin{cases} A = 1 \\ B = 2 \\ C = 3 \\ D = -1 \end{cases}$$

Hence

$$\frac{3s^2+5s+3}{s^4+s^3} = \frac{1}{s} + \frac{2}{s^2} + \frac{3}{s^3} - \frac{1}{s+1}$$

$\Rightarrow$

$$\mathcal{L}^{-1}\left\{\frac{3s^2+5s+3}{s^4+s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= 1 + 2t + \frac{3}{2}t^2 - e^{-t}$$

Here we used the table in §7.2:

$$\mathcal{L}^{-1}\left\{\frac{c}{s}\right\} = c, \quad c \in \mathbb{R}$$

$$\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-k}\right\} = e^{kt}, \quad k \in \mathbb{R}$$

#5: Express the function using window/step functions. Then compute its Laplace transform

$$f(t) = \begin{cases} 0 & t \in [0, \pi/2) \\ \sin t, & t \in [\pi/2, \infty) \end{cases}$$

should "[", not "("

Recall: §7.6. Lecture 17.

step function:

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a. \end{cases}$$

window function:

$$\Pi_{a,b}(t) = \begin{cases} 1 & t \in [a, b) \\ 0. & \text{at other points.} \end{cases}$$

$b > a$

A:

$$f(t) = \sin t \, u(t - \tfrac{\pi}{2})$$

Then we need to compute

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t \cdot u(t - \tfrac{\pi}{2})\}$$

$c = \frac{\pi}{2}$

Recall §7.6 Lecture 17.

$$\rightarrow \quad \mathcal{L}\{g(t-c)\cdot u(t-c)\} = e^{-cs}\,\mathcal{L}\{g(t)\}$$

$G = \mathcal{L}\{g(t)\}$ $\rightarrow \quad \mathcal{L}^{-1}\{e^{-cs}\,G(s)\} = g(t-c)\cdot u(t-c)$

We need to find $g(t)$ such that

$$g(t-\tfrac{\pi}{2}) = \sin t \qquad (*)$$

Let $x = t - \frac{\pi}{2} \Rightarrow t = x + \frac{\pi}{2}$

$$(*) \Rightarrow \quad g(x) = \sin(x+\tfrac{\pi}{2})$$

Hence $g(t) = \sin(t+\frac{\pi}{2})$

$$= \cos t$$

$$\Rightarrow \mathcal{L}\{\underbrace{\sin t}_{g(t-c)}\cdot \underbrace{u(t-\tfrac{\pi}{2})}_{u(t-c)}\}$$

$$= e^{-\frac{\pi}{2}s}\,\mathcal{L}\{g(t)\}$$

$$= e^{-\frac{\pi}{2}s}\,\mathcal{L}\{\cos t\}$$

$$= e^{-\frac{\pi}{2}s}\,\frac{s}{s^2+1}$$

Table in §7.2

$\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$

#6. Solve the I.V.P using Laplace transform. Sketch the graph of the soln.

$$\begin{cases} y'' + y = u(t-2) - u(t-4) \\ y(0) = 1, \quad y'(0) = 0 \end{cases}$$

A:

Recall §7.6 Lecture 17.

① $\mathcal{L}\{u(t-a)\}(s) = \dfrac{e^{-as}}{s}$

② For $b \geq a \geq 0$

$$\mathcal{L}\{\Pi_{a,b}(t)\} = \frac{e^{-as} - e^{-bs}}{s}$$

Here $\Pi_{a,b}(t) = u(t-a) - u(t-b)$

Step 1: Apply Laplace transform $\mathcal{L}$ to the D.E. $\Rightarrow$

$$\mathcal{L}\{y''+y\} = \mathcal{L}\{u(t-2)-u(t-4)\} \quad (\triangle)$$

Write $Y = \mathcal{L}\{y\}$

$$\Rightarrow \mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0)$$

Thm 4 §7.3

$$= s^2 Y - s\cdot 1 - 0$$

$$= s^2 Y - s$$

$$\Rightarrow \text{LHS of } (\triangle) = \mathcal{L}\{y''\} + \mathcal{L}\{y\}$$

$$= s^2 Y - s + Y$$

$$\text{RHS of } (\triangle) = \frac{e^{-2s} - e^{-4s}}{s}$$

$$\Rightarrow s^2 Y - s + Y = \frac{e^{-2s}}{s} - \frac{e^{-4s}}{s}$$

$$\Rightarrow \mathcal{L}\{y\} = Y(s) = \frac{s}{s^2+1} + \frac{e^{-2s}}{s(s^2+1)} - \frac{e^{-4s}}{s(s^2+1)}$$

$$\Rightarrow \quad y = \mathcal{L}^{-1}\{Y\}$$

$$= \underbrace{\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}}_{\cos t} + \underbrace{\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s^2+1)}\right\}}_{?} - \underbrace{\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s(s^2+1)}\right\}}_{?}$$

Table §7.2

Recall: §7.6 Lecture 17

If $F(s) = \mathcal{L}\{f\}(s)$

Then $\mathcal{L}^{-1}\{e^{-cs}F(s)\}(t) = f(t-c)\,u(t-c)$.

$c=2$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{e^{-2s}\cdot \underbrace{\frac{1}{s(s^2+1)}}_{F(s)}\right\}$$

We need to find $f(t)$ s.t

$$F = \mathcal{L}\{f\} \iff f = \mathcal{L}^{-1}\{F\}.$$

Find the partial fractional decomposition of $F = \frac{1}{s(s^2+1)}$: $F = \frac{A}{s} + \frac{Bs+C}{s^2+1}$

$\cdots \Rightarrow A=1,\ B=-1,\ C=0$

Ex

$$F = \frac{1}{s} - \frac{s}{s^2+1}$$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\{F\}(t)$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$= 1 - \cos t$$

$$\Rightarrow \mathcal{L}^{-1}\left\{e^{-2s} \cdot \frac{1}{s(s^2+1)}\right\}$$

$$= f(t-2)\, u(t-2)$$

$$= (1-\cos(t-2))\, u(t-2)$$

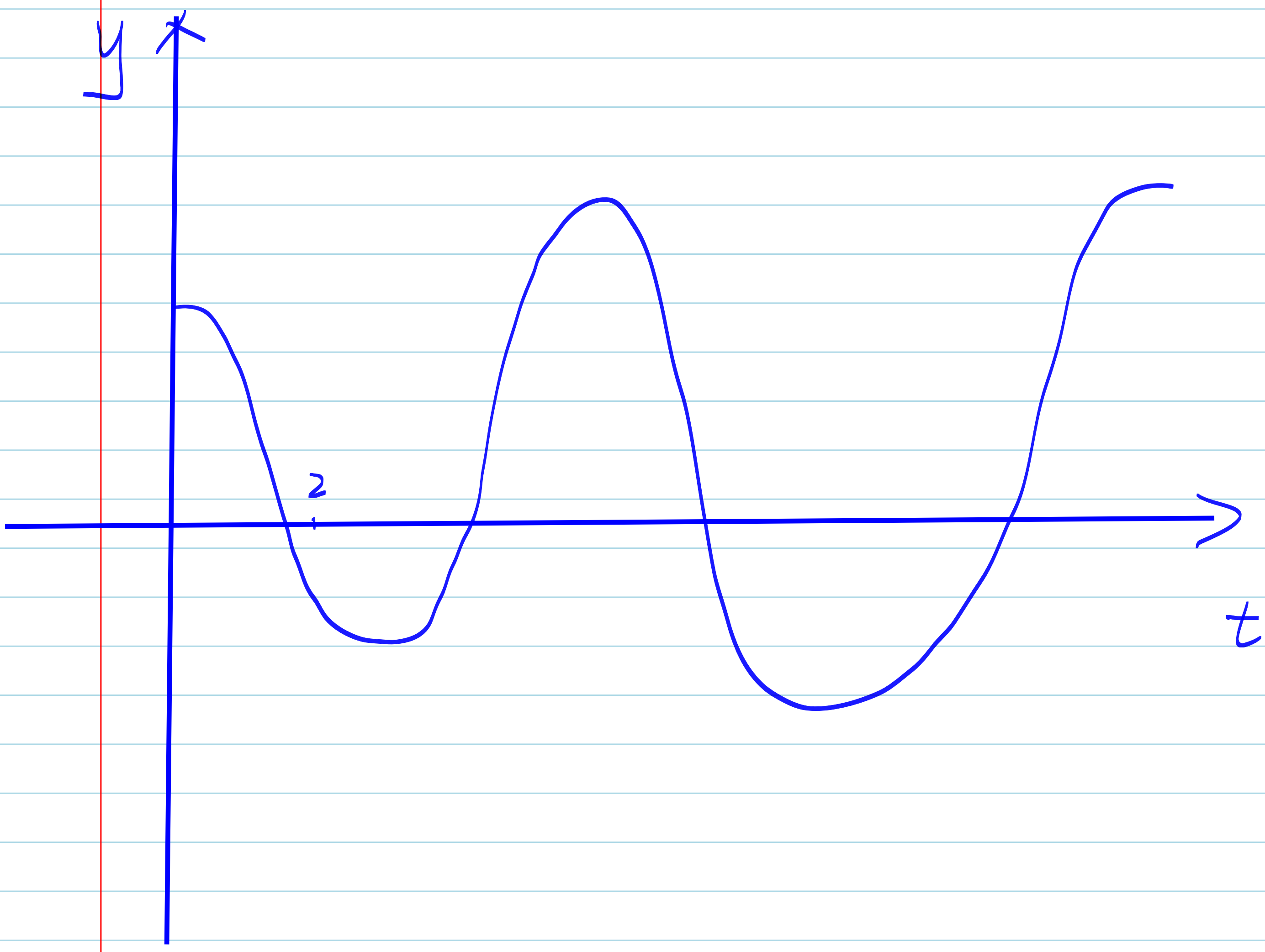
Similarly, $\mathcal{L}^{-1}\left\{e^{-4s} \frac{1}{s(s^2+1)}\right\}$

$$= (1-\cos(t-4))\, u(t-4)$$

$$\Rightarrow$$

$$y = \cos t + (1-\cos(t-2))u(t-2) - (1-\cos(t-4))u(t-4).$$

Graph of y



Note: we will not ask you to plot any graph in the exam.